Numerical Solution of Differential Equations

Here we will consider the numerical solution of differential equations of type

$$\frac{dy}{dx} = f(x,y) \quad \text{— (1)}$$

with initial condition $y = y_1$ at $x = x_1$ — (2)

Here, the function $f(x,y)$ may be a nonlinear function of (x,y) or may be a table of values.

When the value of y is given at $x = x_1$ and the solution is required for $x_1 \le x \le x_f$ then, the problem is called an initial value problem.

If y is given at $x = x_f$ and the solution is required for $x_f \ge x \ge x_1$ then the problem is called a boundary value problem.

The solution of a differential equation says — (means):

A solution ~~cur~~ is a curve $g(x,y)$ in the plane (x,y) whose slope at every point (x,y) in the specified region is given by eqn (1).

The initial point (x_1, y_1) of the solution curve and the slope of the curve at this point are given. Then we extrapolate the values of y for the set of values of x in range $[x_1, x_f]$.

EULER'S METHOD

This method uses the simplest extrapolation techniques to find a solutions.

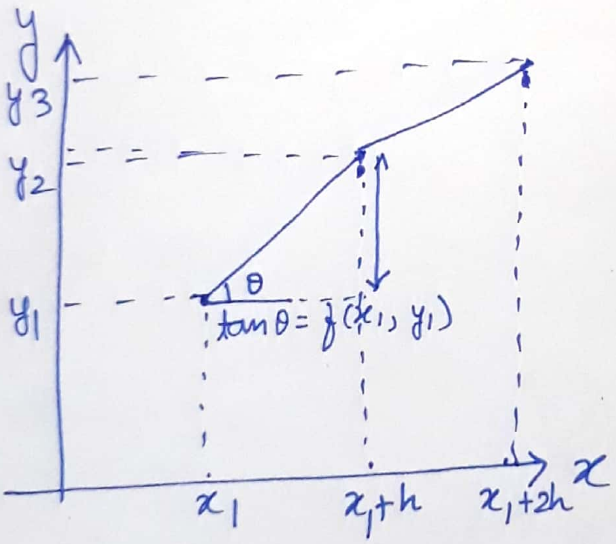
The differential eqn is:

$$\frac{dy}{dx} = f(x, y) \quad \text{———} \; (1)$$

$$y(x_1) = y_1 \quad \text{———} \; (2)$$

Given (x_1, y_1), the slope at this point is obtained by eqn (1), as

$$\frac{dy}{dx}(x_1, y_1) = f(x_1, y_1) \quad \text{———} \; (3)$$



The next point y_2 on the solution curve may be extrapolated by taking a small step (h) in a direction given by the above slope eqn (3).

Therefore;
$$y(x_1+h) = y_2 = y_1 + h\, f(x_1, y_1) \quad \text{———} \; (4)$$

$$\left(\text{as } \tan\theta = f(x_1, y_1) = \frac{(y_2 - y_1)}{(x_1+h) - x_1} = \left(\frac{y_2 - y_1}{h}\right)\right.$$

||ly we can proceed from y_2 to y_3.

In general, the $(i+1)$th point may be obtained from the ith point by

$$y_{i+1} = y_i + h[f(x_i, y_i)] \quad \text{---} \quad (5)$$

Euler's method is basically a technique of developing a piecewise linear approximation of the solution.

At any step the error in the above formula is of the order of $\frac{h^2}{2} y''(z)$, where z is a point in the region of ~~solution~~ computation.

Eg. 1: Solve the differential eqn:

$$\frac{dy}{dx} + xy = 0 \quad , \quad y(0) = 1$$

from $x = 0$ to $x = 0.25$ using Euler's method.

Soln: Let us pick $h = 0.05$, $f(x, y) = -xy$

The Euler's formula,

$$y_{i+1} = y_i + hf(x_i, y_i)$$
$$= y_i + h(-xy) = y_i - hxy$$
$$= y_i - (0.05) x_i y_i$$

given $x_0 = 0$, $y_0 = 1$

$$y_1 = y_0 - (0.05) \times 0 = 1$$
$$y_2 = y_1 - (0.05) x_1 y_1 = 1 - (0.05)(0.05)(1)$$
$$= 0.9975$$

Illy doing like this, we get

x_i	0	0.05	0.1	0.15	0.2	0.25
y_i	1	1	0.997	0.992	0.985	0.975
*Analytical	1	0.999	0.995	0.989	0.980	0.969

Example 9.1
Solve the differential equation

$$\frac{dy}{dx} + xy = 0, \quad y(0) = 1$$

from $x = 0$ to $x = 0.25$ using Euler's method.
Let us pick $h = 0.05$, $f(x, y) = -xy$

The Euler formula is:

$$y_{i+1} = y_i + hf(x_i, y_i) = y_i - hx_iy_i$$
$$= y_i - (0.05)x_iy_i$$

At
$$x_0 = 0,$$
$$y_0 = 1$$
$$y_1 = 1 - 0.05 * 0 = 1$$
$$y_2 = 1 - (0.05)(0.05)(1) = 0.9975$$
$$y_3 = 0.9975 - (0.05)(0.1)(0.9975) = 0.9925$$
$$y_4 = 0.9925 - (0.05)(0.15)(0.9925) = 0.985$$
$$y_5 = 0.985 - (0.05)(0.2)(0.985) = 0.9752$$

The solution is tabulated below:

x_i	0	0.05	0.1	0.15	0.2	0.25
y_i	1	1	0.997	0.992	0.985	0.975

The analytical solution to the differential equation is exp $(-x^2/2)$ and is tabulated below:

x_i	0	0.05	0.1	0.15	0.2	0.25
y_i	1	0.999	0.995	0.989	0.980	0.969

In this example

$$\frac{\partial f}{\partial y} = -x$$

$$\left| 1 + h\frac{\partial f}{\partial y} \right| = |1 - 0.05x| \leq 0.9875 \text{ for } x_{max} = 0.25$$

Thus Euler's method will give a stable solution. The error will however build up as x_i increases. This is clear by inspecting the above tables of solutions obtained by Euler's method and the analytical solution.

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TAYLOR SERIES METHOD:

Consider the following differential equation,

$$\frac{dy}{dx} = f(x, y) \quad \text{——} (1)$$

with $y(x_1) = y_1$ —— (2)

If the solution curve $y(x)$ is expanded in a Taylor series around $x = x_1$, we get

$$y(x) = y_1 + (x-x_1)\frac{dy}{dx} + \frac{(x-x_1)^2}{2}\frac{d^2y}{dx^2} + \frac{(x-x_1)^3}{6}\frac{d^3y}{dx^3} + \cdots \quad \text{——} (3)$$

$\Rightarrow$ y at $x = x_1 + h$ can be obtained from the series,

$$y(x_1+h) = y_1 + h\left.\frac{dy}{dx}\right|_{x_1} + \frac{h^2}{2}\left.\frac{d^2y}{dx^2}\right|_{x_1} + \frac{h^3}{3}\left.\frac{d^3y}{dx^3}\right|_{x_1} + \cdots \quad \text{——} (4)$$

Observe that $\frac{dy}{dx} = f(x,y)$ and that y is a function of x. Thus,

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d}{\partial x} f(x,y)$$

$$= \frac{\partial}{\partial x} f(x,y) + \frac{\partial}{\partial y} f(x,y)\frac{dy}{dx}$$

Let $\frac{\partial}{\partial x} f(x,y) = f_x$, $\frac{\partial}{\partial y} f(x,y) = f_y$

and $f(x,y) = f$ —— (5)

Eq^n (4) can be expressed as

$$y(x_1+h) = y_1 + h f(x_1, y_1) + \frac{h^2}{2}\left[f_x + f(x_1,y_1) f_y\right] + \cdots \quad \text{——} (6)$$

from eqn ⑥, we can observe that Euler's method of extrapolation corresponds to taking only the first two terms of the above series.

So the errors due to the truncation of the series would be of the order of h^2.

An improvement of the Euler method would thus be to include the h^2 term in the above expansion and use the formula ⑦ to extrapolate to the point (x_1+h) from x_1,

$$y_2 = y_1 + h f(x_1, y_1) + \frac{h^2}{2}\left[f_x(x_1, y_1) + f(x_1, y_1) f_y(x_1, y_1) \right] \quad ⑦$$

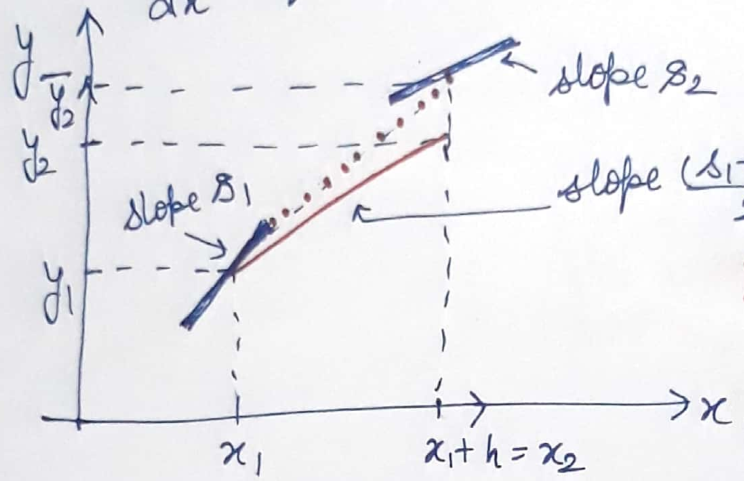
In above formula the truncation error is of the order of h^3.

But this method is not generally applicable as finding $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$ are not easy to find.

[RUNGE-KUTTA II order method] HEUN'S METHOD

Consider the following geometric method of extrapolating the $y(x)$ curve to find the solution of eqn:

$$\frac{dy}{dx} = f(x,y), \quad \text{and} \quad y(x_1) = y_1 \qquad (1)$$



Heun's Method

RUNGE-KUTTA second order Method

Here a straight line is drawn from point (x_1, y_1) with a slope $s_1 = f(x_1, y_1)$. Let it cut the vertical line through x_1+h at $(x_1+h, \bar{y}_2)$.

Now find slope $\frac{dy}{dx}$ of the solution curve $y(x)$ at this point $(x_1+h, \bar{y}_2)$ given by $s_2 = f(x_2, \bar{y}_2)$.

Now draw a straight line from (x_1, y_1) with a slope $(s_1+s_2)/2$. The point y_2 where this straight line cuts the vertical line at (x_1+h) is the approximate solution of the differential equation at x_1+h.

So
$$y_2 = y_1 + h\frac{(s_1+s_2)}{2} \qquad (2)$$

In general, the $(i+1)^{th}$ point is obtained from ith point,

$$y_{i+1} = y_i + \frac{h}{2}(s_i + s_{i+1}) \qquad (3)$$

with $s_i = f(x_i, y_i)$ and $s_{i+1} = f(x_{i+1}, y_i + s_i h)$.

This method is called second order Runga-kutta method (Heun's method).

Eqn ③ gives a formula which uses ~~only~~ function $f(x, y)$ at (x_i, y_i) and at $(x_i, y_i + s_i h)$.

Algorithm:

1. Read x_1, y_1, h, x_f
2. While $x_1 \le x_f$ do

   begin

3. Write x_1, y_1
4. $s_1 \leftarrow f(x_1, y_1)$
5. $x_2 \leftarrow x_1 + h$
6. $y_2 \leftarrow y_1 + h s_1$
7. $s_2 \leftarrow f(x_2, y_2)$
8. $y_2 \leftarrow y_1 + h \times (s_1 + s_2)/2$
9. $x_1 \leftarrow x_2$
10. $y_1 \leftarrow y_2$

    end

11. stop

Example:

solve the differential equation,

$$\frac{dy}{dx} + xy = 0 , \quad x=0 , \; y(0)=1$$

By Heun (Runge-Kutta II order) formula,

$$y_{i+1} = y_i + \frac{h}{2}(s_1+s_2)$$

$$s_1 = f(x_i, y_i) = \frac{dy}{dx}(x_i, y_i)$$

$$s_2 = \frac{dy}{dx}(x_i+h, \; y_i + s_1 h)$$

$$h=0.05, \quad x_f = 0.25 , \quad x_0 = 0 , \quad y_0 = 1$$

At $x=0$, $s_1 = \frac{dy}{dx} = -xy = 0$

At, $x=0.05$, $s_2 = -(0.05)(1) = -0.05$

$$(s_1+s_2)/2 = -0.025$$

$$y_1 = y_0 + h(s_1+s_2)/2$$

$$= 1 + (0.05)(-0.025) = 0.9988 \simeq 0.999$$

$$s_1 = \frac{dy}{dx}(x_1, y_1) = -(0.05)(0.9988) = -0.04994$$

$$s_2 = \frac{dy}{dx}(x_2, \; y_1 + s_1 h) = -(0.15)(0.9963)$$

$$= -0.1494$$

$$(s_1+s_2)/2 = -0.0996\,7$$

$y_2 = y_1 + h \times 0.09967$

$= 0.9988 - (0.05)(0.09967)$

$= 0.9938 \simeq 0.994$

$y_3 = y_2 + h \times \frac{(s_1+s_2)}{2}$

$= 0.9938 - (0.05)(0.1205) = 0.988$

$y_4 = y_3 + h(s_1+s_2)/2$

$= 0.988 - (0.05)(0.1722) = 0.979$

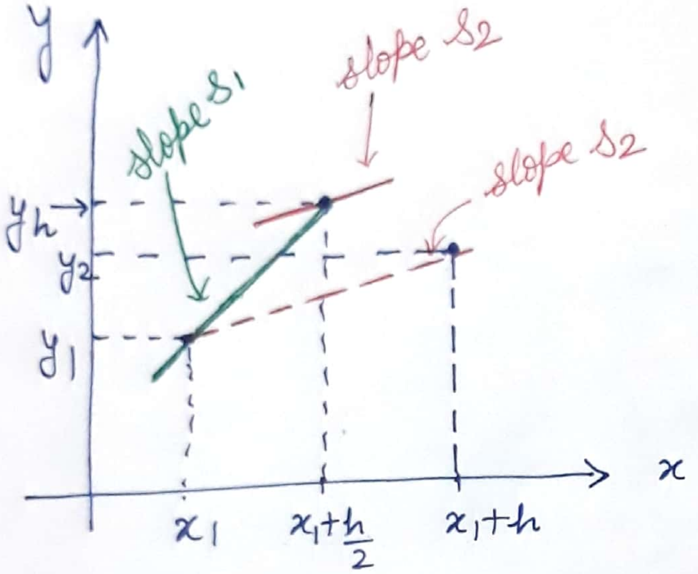
$y_5 = 0.979 - (0.05)(0.2192) = 0.968$

methods ↓	$x_i \rightarrow$	0	0.05	0.1	0.15	0.20	0.25
Heun's	y_i ↓	1	0.999	0.994	0.988	0.979	0.968
- Euler's (Page no. 3)	↓	1	1	0.997	0.992	0.985	0.975
Analytical		1	0.999	0.995	0.989	0.980	0.969

We can see that Heun's method sol^n is much closer to the analytical solution and the error does not grow.

POLYGON METHOD (Second order RUNGE-KUTTA method)

In this method we use value of the function $f(x,y)$, not any of its partial derivatives.



We start a line with slope $s_1 = f(x_1, y_1)$ from point (x_1, y_1).

The point where this line cuts the vertical line erected at $(x_1+\frac{h}{2})$ gives the value of $y = y_h$.

Now, we calculate $f(x_1+\frac{h}{2}, y_h)$ which will give the slope of the solution curve at this point.

Again drawing a straight line from (x_1, y_1) with slope s_2, this cuts the vertical line erected at (x_1+h) at y_2. This is taken as the approximate solution of the differential equation at (x_1+h).

In general, we can write

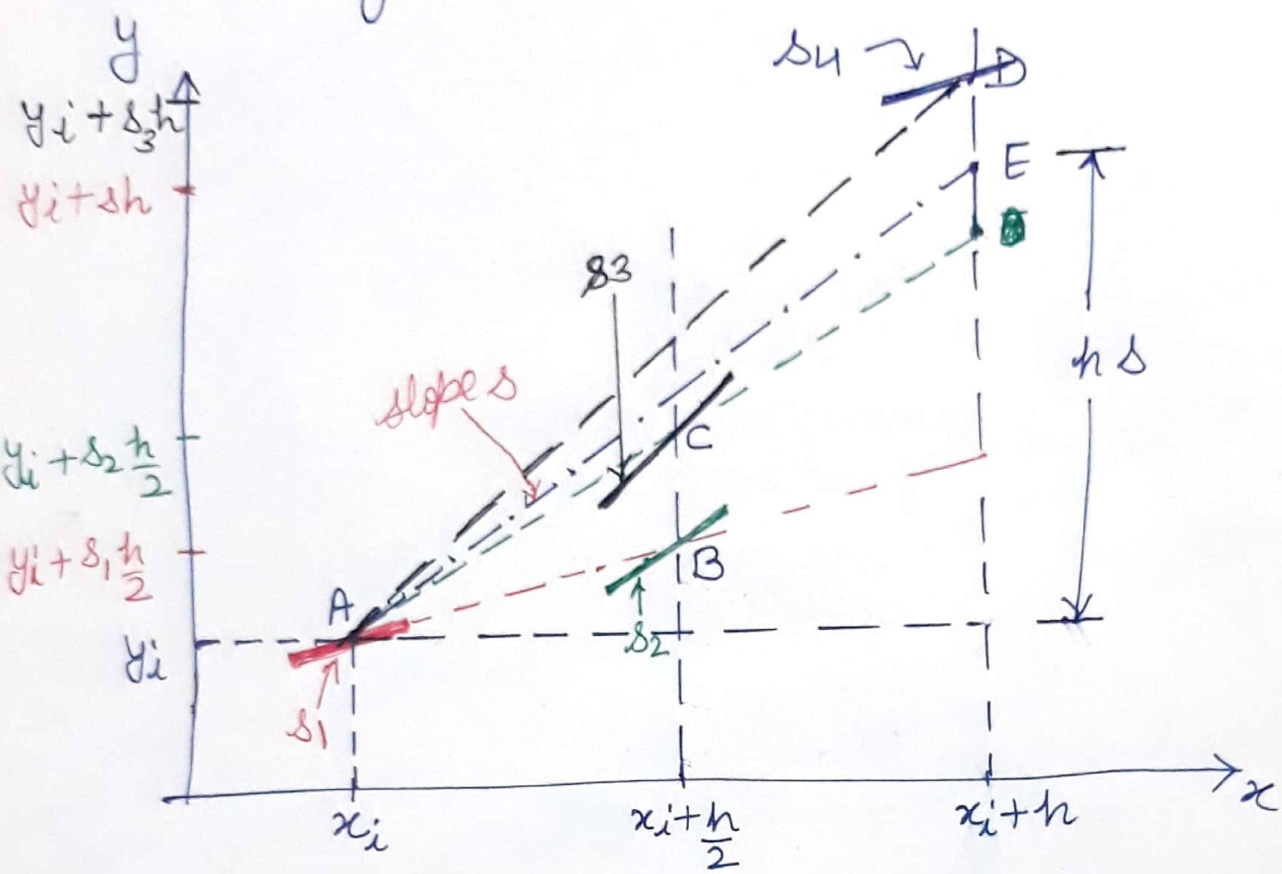
$$y_{i+1} = y_i + h f\left(x_i+\frac{h}{2},\ y_i + s_i\frac{h}{2}\right) \quad \text{—— (1)}$$

where $\quad s_i = f(x_i, y_i)$

This method is exactly equivalent to the Taylor series method upto second degree terms in h.

RUNGE-KUTTA fourth oder method

Geometrically this method can be explained below.



We draw a straight line from point $A(x_i, y_i)$, with slope $s_1 = f(x_i, y_i)$. Let this straight line cut the vertical line erected at $(x_i + \frac{h}{2})$ at B.

Now find the slope s_2 of the solution curve at B. This slope s_2 equals $f(x_i + \frac{h}{2}, y_i + s_1 \frac{h}{2})$. Let this line of slope s_2 starting from (x_i, y_i) cut the vertical line at $x_i + \frac{h}{2}$ at C.

Find the slope s_3 at point C. This slope s_3 equals $f(x_i + \frac{h}{2}, y_i + s_2 \frac{h}{2})$. Now, draw another straight line starting at A with slope s_3. Let this cut vertical line erected at $x_i + h$ at D.

Find the slope s_4 of the solution curve at D.

This slope $s_4 = f(x_i + h,\ y_i + s_3 h)$.

Find a weighted average of the four slopes,

$$s = \frac{1}{6}(s_1 + 2s_2 + 2s_3 + s_4) \quad \text{—— (1)}$$

Now, draw a straight line from point A with this slope s given by eqn (1). Let it cut the vertical line erected at $(x_i + h)$ at E.

The point E is taken as the solution of the differential eqn at $(x_i + h)$.

Or,

$$y_{i+1} = y_i + \frac{h}{6}(s_1 + 2s_2 + 2s_3 + s_4) \quad \text{—— (2)}$$

where

$$s_1 = f(x_i, y_i) \quad \text{—— (3)}$$

$$s_2 = f\left(x_i + \frac{h}{2},\ y_i + s_1\frac{h}{2}\right) \quad \text{—— (4)}$$

$$s_3 = f\left(x_i + \frac{h}{2},\ y_i + s_2\frac{h}{2}\right) \quad \text{—— (5)}$$

$$s_4 = f(x_i + h,\ y_i + s_3 h) \quad \text{—— (6)}$$

This formula matches the Taylor series upto the h^4 term and thus has a truncation error of the order of h^5.

Algorithm 9.2 Solution of Differential Equation by Runge-Kutta Method

```
1  Read x1, y1, h, xf
2  while x1 ≤ xf do
      begin
3        Write x1, y1
4        s1 ← f(x1, y1)
5        x2 ← x1 + 0.5h
6        y2 ← y1 + 0.5hs1
7        s2 ← f(x2, y2)
8        y2 ← y1 + 0.5hs2
9        s3 ← f(x2, y2)
10       x2 ← x1 + h
11       y2 ← y1 + s3h
12       s4 ← f(x2, y2)
13       s ← (s1 + 2s2 + 2s3 + s4)
14       y2 ← y1 + (hs/6)
15       x1 ← x2
16       y1 ← y2
      end
17 Stop
```